SUBORBITAL GRAPHS OF THE CYCLIC GROUP C_n ACTING ON VERTICES OF A REGULAR $n-$GON

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ABSTRACT

Let G be the cyclic group $C_n = < x > = < (12 \ldots n) >$ *acting on a set* $X = \{1, 2, \ldots, n\}$*; the set of vertces of a regular* $n - gon$*. In this paper it is shown that the action is transitive, and if the degree n is not prime then the action is imprimitive. The rank of G on X is shown to be n and the corresponding subdegrees are* $1, 1, 1, \ldots, 1$*; nones. It is also proven that the suborbit* Δ_i *of G is paired with* Δ_{n-i}*. Further it is shown that for a suborbital* O_{i-1} *in G,* $(a, b) \in O_{i-1}$ *if and only if* $|b - a| = \begin{cases} i - 1 & if b > a \\ n - (i - 1) & if a > b \end{cases}$*, and that all suborbital graphs of G are connected if and only if n is prime. Finally it is shown that the number of components of the suborbital graph* Γ_{i-1} *is* $d = gcd(n, i - 1)$ *and its girth is* $r = \frac{n}{d}$*, when* $d \neq \frac{n}{2}$ *and zero if* $d = \frac{n}{2}$ *.*

Key Words: cyclic group, primitive action, suborbits, subdegrees, ranks and suborbital graphs.

1. Introduction

Definition 1.1 A dihedral group is the group of symmetries of a regular polygon. It is denoted by D_n where $n \geq 3$, and has an order $2n$. The conventionally way of writing $Dn = < x, y \mid x^n = y^2 = 1, yx = x^{n-1}y = x^{-1}y >$, thus D_n is the group generated by the elements x, y subject to the conditions $x^n = y^2 = 1$; $yx = x^{n-1}y = x^{-1}y$. Here $x$is a rotation about the center of the polygon through angle $\frac{2\Pi^c}{n}$; it generates a cyclic subgroup C_n of order *n* which is being considered in this paper.

2. Preliminary definitions and results

Definition 2.1 The action of a group G on a set $X$is said to be transitive if for each pair of points $x, y \in X$, there exist $g \in G$ such that $gx = y$; in other words the action has only one orbit.
Let G be a transitive permutation group acting on a set X. If G_x or $Stab_G(x)$ is the stabilizer of $x \in X$. The orbits $\Delta_0 = \{x\}, \Delta_1, \Delta_2, \ldots, \Delta_{r-1}$ of G_x on X are known as suborbits of the group G. In this case rank of G is equal to r which is the number of suborbits of G on X. The lengths of the orbits of G_x on $X$are called the subdegrees of G. It was shown in [3] that both the cardinalities of suborbits Δ_i $(i = 0, 1, 2, \ldots, r - 1)$ and the rank are independent of the choice of $x \in X$.
Theorem 2.2*(Orbit - Stabilizer Theorem [5], p. 72)*
Let G be a group acting on finite set X and $x \in X$. Then

$$|Orb_G(x)| = |G : Stab_G(x)|.$$

Theorem 2.3*(See [2], p. 98)*
Let G be a finite group acting on a set X. The number of orbits of G is

$$\frac{1}{|G|}\sum_{g\in G}|fix(g)|$$

(This theorem is referred to as Cauchy - Frobenius Lemma).

Theorem 2.4 *(See [4])*
Let G be a transitive permutation group acting on a set X and let $x \in X$. Then G is primitive if and only if G_x is a maximal subgroup or equivalently G is imprimitive if and only if G_x is not a maximal subgroup of G.

Theorem 2.5*(See [6])*
If G is acts on a set X, where G is a transitive group of prime degree, then G is primitive.

Theorem 2.6*(see [1], p.422)*
If G is primitive, with subdegrees $1 = n_0, n_1, \ldots, n_{r-1}$(in increasing order of magnitude), then $n_1 n_{i-1} \geq n_i$ for $i = 1, \ldots, r-1$. Now if there exist an index $i > 0$ such that $n_i > n_1 n_{i-1}$, then G is imprimitive.

Definition 2.7 Let $\triangle$ be an orbit of G_x. Define $\triangle* = \{gx|g \in G, x \in g\triangle\}$, then $\triangle*$ is also an orbit of G_x and is called the G_x-orbit (or the G - suborbit) paired with $\triangle$. Clearly $|\triangle| = |\triangle*|$. If $\triangle* = \triangle$, then $\triangle$ is called a selfpaired orbit of G_x.

Theorem 2.8 *(See [1])*
Let G act transtively on a set X and let $g \in G$, then the number of selfpaired suborbits of G is given by

$$\frac{1}{|G|}\sum_{g\in G}|fix(g^2)|.$$

Theorem 2.9 *(See [8], Section 16.5)*
G_x has an orbit different from $\{x\}$ and paired with itself if and only if Ghas even order.

Definition 2.10 Observe that G acts on $X \times X$ by $g(x,y) = (gx, gy)$, $g \in Gx, y \in X$. If $O \subseteq X \times X$ is a G - orbit on $X \times X$, then for a fixed $x \in X$, $\triangle = \{y \in X \mid (x, y) \in O\}$ is a $G_x - orbit$ on X. Conversely, if $\triangle \subseteq X$ is a $G_x -$orbit, then $O = \{(gx, gy) \mid g \in G, y \in \triangle\}$ is a $G-orbit$ on $X \times X$. We say that $\triangle$ corresponds to O.The G - orbits on $X \times X$ are called suborbitals. Let $O_i \subseteq X \times X$, $i = 0, 1, 2, \ldots, r-1$ be a suborbital. Then we form a graph Γ_i, by taking X as the set of vertices of Γ_i, and by including a directed edge from $x$to $y (x, y \in X)$ if and only if $(x, y) \in O_i$.
The suborbital graph Γ_0 corresponding to the suborbit $\triangle_0$ is called the trivial suborbital graph. When the suborbits are selfpaired the corresponding suborbital graphs are undirected. If the suborbits are not selfpaired the corresponding suborbital graphs are directed. The trivial suborbital graph is selfpaired; it consists of a loop

based at each vertex $x \in X$. We are mainly interested with the non - trivial suborbital graphs. If the suborbital graph Γ is paired with Γ^*, then Γ^* is just Γ with arrows reversed.

Theorem 2.11*(See [7])*
Let G be transitive on X. Then G is primitive if and only if each suborbital graph $\Gamma_i (i = 1, 2, \ldots, r - 1)$ is connected.

3. Main results

3.1Stabilizer, transitivity and primitivity of *G* acting on *X*

From here hence forth $G = C_n = < x > = < (12 \ldots n) >$ and $X = \{1, 2, \ldots, n\}$; the set of vertices of a regular $n-$ gon.

Theorem 3.1*Let $i \in X$, then $Stab_G(i) = \{1\}$.*
Proof. Clearly in *G*, it is only the identity element which fixes a point in *X*.

Theorem 3.2 *G acts transitively on X.*
Proof. From Theorem 3.1, only the identity element which has a fixed point in *X*and in this case the number of points fixed by the identity is $|X| = n$. Hence by Cauchy - Frobenius lemma the number of orbits of *G* on *X* is

$$\frac{1}{|G|}\sum_{g\in G} |fix(g)| = (1)/(n) \times n = 1$$

Therefore from definition 2.1, *G* acts transitively on *X*.

Theorem 3.3. If $|X| = n$, where n is not a prime number, then G acts imprimitively on X.
Proof. Since n is not prime, then there exists a positive integer $k$such that $1 < k < n$ and k divides n. Now $< x^{n/k} >$ is a proper subgroup of G of order k properly containing $Stab_G(i) = \{1\}$. Hence by Theorem 2.4G acts imprimitively on X.

3.2 Ranks, suborbits and subdegrees of *G*

Theorem 3.4. Orbits of $Stab_G(1)$ on X are $\Delta_0 = \{1\}, \Delta_1 = \{2\}, \Delta_2 = \{3\}, \ldots, \Delta_i = \{i + 1\}, \ldots, \Delta_{n-1} = \{n\}$. Thus the rank of G on X is $n$and the subdegrees are $1,1,1, \ldots, 1; n$ones.
Proof. From Theorem 3.1$Stab_G(1) = \{1\}$, and therefore suborbits of G consist only of singleton elements. Hence the rank of G is n and the subdegrees are $1,1,1, \ldots,1; n$ ones

Theorem 3.5. *The number of selfpaired suborbits of G on X is 2 if n is even or 1 if n is odd.*
Proof. Let $g \in G$, then g^2 will have fixed points in X if either g is the identity or g is an element of order two. Also G contains an element of order two only when$n$ is even. Therefore by Theorem 2.8 the number of selfpaired suborbits of G is $\frac{1}{n}(n + n) = \frac{2n}{n} = 2$ when n is even and $\frac{1}{n}(n) = \frac{n}{n} = 1$ when n is odd.

Theorem 3.6. *Let $G = C_n$ act on X, then the suborbit Δ_i of G is paired with Δ_{n-i}.*
Proof. Let $G = <x>$ and $i + 1 \in \Delta_i$ (see Theorem 3.4.). To get the suborbit paired with Δ_i, first find $x^j \in G$ where $0 \leq j \leq n$ such that $x^j(i + 1) = 1$. The value of $j$is gotten by solving the equation $(j + i + 1) mod n = 1$, which can be rewritten in this case as

$$j + i + 1 = n + 1$$
$$j = n - i$$

Secondly find where x^j takes 1 i.e $x^j 1$, which $is j + 1 = n - i + 1$. By Definition 2.7, the element $n - i + 1$ exist in the suborbit which is paired with Δ_i. If $i + 1 \in \Delta_i$ then $n - i + \in \Delta_{n-i}$. Hence the suborbit Δ_i is paired with the suborbit Δ_{n-i}, this is $\Delta_i^* = \Delta_{n-i}$.

Corollary 3.7.Δ_0*is the only selfpaired suborbit of G when n is odd and when n is even Δ_0 and $\Delta_{\frac{n}{2}}$ are selfpaired suborbits.*
Proof. From Theorem 3.6, $\Delta_0^* = \Delta_{n-0} = \Delta_n = \Delta_0$ and $\Delta_{\frac{n}{2}}^* = \Delta_{n-\frac{n}{2}} = \Delta_{\frac{n}{2}}$.

3.3 Suborbital graphs for *G* acting on *X*

The suborbitals in this section have a one to one correspondence with the suborbits of G in Section 3.2. So Δ_i corresponds to O_i. Elements in X are assumed to be arranged cyclically and evenly spaced around a circle in anticlockwise direction. Any element $x^k \in G$ takes $i \in X$, k units around the circle in an anticlockwise direction.
Theorem 3.8. Suppose $(1, i)$is a representative of the non-trivial suborbital O_{i-1} of G, then $(a, b) \in O_{i-1}$ if and only if

$$|b - a| = \begin{cases} i - 1 & if \ b > a \\ n - (i - 1) if a > b \end{cases}$$

Proof. Suppose $(a, b) \in O_{i-1}$, where $i > 1$, then there exists $x^j \in G$ such that $x^j(1, i) = (a, b)$. Now if $b > a$, then $a = 1 + j$and $b = i + j$. Thus $|b - a| = i - 1$. Next if $a > b$, then $i + j > n$; and $a = 1 + j$ and $b = i + j - n$ implies $a - b = n - (i - 1)$. Therefore $|b - a| = n - (i - 1)$.
Conversely, suppose that

$$|b - a| = \begin{cases} i - 1 & if \ b > a \\ n - (i - 1) if a > b \end{cases}$$

We need to show that $(a, b) \in O_{i-1}$. In other words we need to show that there exists $x^k \in G$ such that $x^k(1, i) = (a, b)$. Now if $b > a$, $b - a = i - 1$ implies $a = b - i + 1$. Therefore

$$x^{a-1}(1, i) = (a, a - 1 + i) = (a, b).$$

On the other hand if $a > b$, then $a - b = n - (i - 1)$ implies $a = n + b - i + 1$and

$$x^{a-1}(1, i) = (a, a + i - 1) = (a, n + b) \equiv (a, b)(mod n).$$

Theorem 3.9. *Elements of O_{i-1} can be obtained by pairing each point of $x^{i-1} \in G$ to a point it is being mapped to.*
Proof. Consider $O_{i-1}(1,i) = \{(1,i),(2,i+1), \dots, (k,k+i-1), \dots, (n,i-1)\}$ and a rotation $x^{i-1} \in G$, where $1 < i \leq n$. In the rotation

$$x^{i-1} = \begin{pmatrix} 1 & 2 & \dots k \dots n \\ (1+i-1) & (2+i-1) & \dots (k+i-1) \dots (n+i-1) \end{pmatrix},$$

if each point is paired with the point it is mapped to, we obtain

$$\{(1,1+i-1),(2,2+i-1), \dots, (k,k+i-1), \dots, (n,n+i-1)\}$$
$$= \{(1,i),(2,i+1), \dots, (k,k+i-1), \dots, (n,i-1)\} = O_{i-1}.$$

From Theorem3.9 we can deduce the following results.

Corollary 3.10.*There is a one to one correspondence between the cycles of x^{i-1} and the cycles of the suborbital graph Γ_{i-1}.*

Corollary 3.11.*The number of components of the suborbital graph Γ_{i-1} is equal to $gcd(n,i-1) = d$, and its girth is $r = \frac{n}{d}$, where $n,d,r,i \in \mathbb{Z}$ and $d \neq \frac{n}{2}$. When $d = \frac{n}{2}$, then the girth is zero.*
Proof. The number of disjoint cycles of $x^{i-1} \in G$ is equal to $gcd(n,i-1) = d$, and all the cycles are of equal length, which is $r = \frac{n}{d}$. From Corollary 3.10 we deduce that Γ_{i-1} has d components each of which is a cycle of length$r = \frac{n}{d}$ and therefore the girth of Γ_{i-1} is $\frac{n}{d}$ when $d \neq \frac{n}{2}$. When $d = \frac{n}{2}$, then $r = 2$, but since Γ_{i-1} does not have multiple edges, the girth of Γ_{i-1}must be zero in this case.

Corollary 3.12.T*he number of connected suborbital graphs is $\varphi(n)$, whereφ is the Euler's phi function.*
Proof. Since $\varphi(n)$ is the number of i, $1 \leq i \leq n$ such that $gcd(n,i) = 1$, then from corollary 3.11. the number of suborbital graphs of *G* with exactly one connected component is $\varphi(n)$.
From Definition 2.9, Theorem 3.6 and Corollary 3.7 the following two results follow.

Theorem 3.13. The suborbital graphs $\Gamma_0$and $\Gamma_{\frac{n}{2}}$are undirected when *n* is even and the other non - trivial suborbital graphs are directed .

Theorem 3.14. When n is odd only the trivial suborbital graph Γ_0 is undirected and the other non - trivial graphs are directed.
From Theorems 2.5, Theorems 2.10 and 3.3 the following result is trivial.

Theorem 3.15. All the non - trivial suborbital graphs of *G* are connected if and only if *n* is prime

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