RANKS AND SUBDEGREES OF THE DIRECT PRODUCT OF THE ALTERNATING GROUP ACTING ON THREE SETS

Author: Lewis Nyaga

Department of Pure and Applied Mathematics

Jomo Kenyatta University of Agriculture and technology

P O BOX 62000 00200

Nairobi, Kenya

Email: lnyaga@jkuat.ac.ke

Abstract

Ranks and sub degrees of the action of the direct product of the alternating Group on Cartesian product of three sets is investigated. In this paper, we show that if $n \geq 4$*, the rank of* $A_n \times A_n \times A_n$ *acting on three disjoint set is equal to 8 and the sub degrees are; 1, n – 1,* $(n-1)^2$*,* $(n-1)^3$*.*

Keyword: Ranks, Sub degrees, Direct Product, Alternating Group, Cartesian product

1. Introduction

In this paper we consider the Alternating groups (A_n, X_1), (A_n, X_2) and (A_n, X_3), where the sets X_1, X_2 and X_3 are disjoint and of cardinality *n*. So the direct product $A_n \times A_n \times A_n$ acts on the Cartesian product $X_1 \times X_2 \times X_3$ by the rule

$$(x_1, x_2, x_3)(g_1, g_2, g_3) = (x_1 g_1, x_2 g_2, x_3 g_3) \forall x_i \in X_i, g_i \in A_i$$

We shall calculate the ranks and sub degrees of $A_2 \times A_2 \times A_2$, $A_3 \times A_3 \times A_3$, $A_4 \times A_4 \times A_4$ and $A_5 \times A_5 \times A_5$ before giving the results for $A_n \times A_n \times A_n$

2. Notation and Preliminary results

Let *G* act on a set *X*. Then *X* is partitioned into disjoint equivalence classes called orbits or transitivity classes of the action. For each *x* ∈*X* the orbit containing *x* is called the orbit of *x* and is denoted by $Orb_G x$. Thus $Orb_G x = \{gx \mid g \in G\}$.

The action of a group *G* on the set *X* is said to be transitive if for each pair of points *x*, *y* ∈*X*, there exists *g* ∈*G* such that $gx = y$; in other words, if the action has only one orbit.

Let *G* act on a set *X* and let *x* ∈*X*. The stabilizer of *x* in *G* is denoted by $Stab_G x$ is given by $Stab_G x = \{g \in G \mid gx = x\}$. $Stab_G x$ forms a subgroup of *G* called the Isotropy group of *x*. It is also denoted by G_x.

Let *G* act on a set *X*. The set of elements of *X* fixed by *g* ∈*G* is called the fixed point set of *G* and is denoted by $Fix(g)$. Thus $Fix(g) = \{x \in X \mid gx = x\}$.

Theorem 2.1 Let G be a finite group acting on a set *X*. The number of orbits in *X* under *G* is given by $\frac{1}{|G|}\sum_{g \in G} |fix(g)|$ (Rose, 1978; Gardiner, 1986; Flareigh, 1994).

Theorem 2.2 Let *G* be a group acting on a finite set *X* and *x*∈*X*. Then $|Orb_G x| = |G:G_x|$, the index of G_x in *G*. (Rose, 1978; Gardiner, 1986; Flareigh, 1994).

Theorem 1.2 is called the *Orbit-Stabilizer Theorem.*

Suppose *G* is a group acting transitively on a set *X* and let G_x be the stabilizer in *G* of a point *x*∈*X* . The orbits $\Delta_0 = x, \Delta_1, \Delta_2, ..., \Delta_{r-1}$ of G_x on *X* are known as suborbits of *G*. The rank of *G* in this case is *r*. The sizes $n_i = |\Delta_i| (i = 0, 1, ..., r-1)$ often called the ’lengths’ of suborbits are known as the subdegrees of *G*. It can be shown that both r and the cardinalities of the suborbits. $\Delta_i (i = 0,1,...,r-1)$ are independent of the choices of $x \in X$.(

Let (G_1, X_1) and (G_2, X_2) be permutation groups. The direct product $G_1 \times G_2$ acts on the disjoint union $X_1 \cup X_2$ by the rule

$$x(g_1, g_2) = \begin{cases} xg_1, \text{ if } x \in X_1 \\ xg_2, \text{ if } x \in X_2 \end{cases},$$

and on the Cartesian product $X_1 \times X_2$ by the rule $(x_1, x_2)(g_1, g_2) = (x_1 g_1, x_2 g_2)$ (Cameron *et al.*, 2008)

3. Main results

Lemma 3.1. If (A_2, X), (A_2, Y) and (A_2, Z) are alternating groups with $X = \{x_1, x_2\}, Y = \{y_1, y_2\}$ and $Z = \{z_1, z_2\}$, then the rank of $A_2 \times A_2 \times A_2$ acting on $X \times Y \times Z$ is 8 and the sub degree of each suborbit is 1.

Proof. Let $G = A_2 \times A_2 \times A_2$ and $K = X \times Y \times Z$, then the elements of G are (e_1, e_2, e_3) where each e_i represents the identity from each alternating group. The stabilizer of an element (x_1, y_1, z_1) is $stab_G(x_1, y_1, z_1) = (e_1, e_2, e_3)$ and since the identity fixes all the elements, the total number of elements in K fixed by each permutation in G is 8. Hence by Theorem 2.1, the number of orbits of $G_{(x_1, y_1, z_1)}$ acting on K are given by

$$\frac{1}{\left|G_{(x_1, y_1, z_1)}\right|} \sum_{g_1, g_2, g_3 \in G_{(e_1, e_2, e_3)}} \left|Fix(g_1, g_2, g_3)\right| = \frac{1}{1}(1 \times 8) = 8$$

Each element in K is in its own suborbits since multiplying the elements with the identity permutation fixes every element, thus each suborbit has a single element making the subdegrees to be 1.

Lemma 3.2. If (A_3, X), (A_3, Y) and (A_3, Z) are alternating groups with $X_1 = \{x_1, x_2, x_3\}, Y = \{y_1, y_2, y_3\}$ and $Z = \{z_1, z_2, z_3\}$, then the rank of $A_3 \times A_3 \times A_3$ acting on $X \times Y \times Z$ is 27 and the length of each suborbit is 1.

*Proof.*The elements of (A_3, X), (A_3, Y) and (A_3, Z) are $(e_1, (x_1x_2x_3), (x_1x_3x_2))$, $(e_2, (y_1y_2y_3), (y_1y_3y_2))$, and $(e_3, (z_1z_2z_3), (z_1z_3z_2))$ respectively and therefore, $G = A_3 \times A_3 \times A_3$ has 27 elements from the direct product, and if $K = X \times Y \times Z$, then K has 27 elements each of which is an ordered triple. $stab_G(x_1, y_1, z_1) = (e_1, e_2, e_3)$ and since the identity fixes all the elements, the total number of elements in K fixed by each permutation in G is 27. Hence by Theorem 2.1, the number of orbits of $G_{(x_1, y_1, z_1)}$ acting on K are given by

$$\frac{1}{\left|G_{(x_1, y_1, z_1)}\right|} \sum_{g_1, g_2, g_3 \in G_{(x_1, y_1, z_1)}} \left|Fix(g_1, g_2, g_3)\right| = \frac{1}{1}(1 \times 27) = 27$$

Each element in K is in its own suborbits since multiplying the elements with the identity permutation fixes every element, thus each suborbit has one element and thus making the sub degree to be 1.

Lemma 3.3. If (A_4, X), (A_4, Y) and (A_4, Z) are alternating groups with $X = \{x_1, x_2, x_3, x_4\}, Y = \{y_1, y_2, y_3, y_4\}$ and $Z = \{z_1, z_2, z_3, z_4\}$, then the rank of $A_4 \times A_4 \times A_4$ acting on $X \times Y \times Z$ is 8 and the subdegrees are 1, 3, 9, 27.

Proof. Let $G = A_4 \times A_4 \times A_4$ and $K = X \times Y \times Z$. If (x_1, y_1, z_1) represents an arbitrary element from K, then

$$\begin{aligned}stab_G(x_1, y_1, z_1) = \{&(e_1, e_2, e_3), (e_1, (y_2y_3y_4), e_3), (e_1, (y_2y_4y_3), e_3), ((x_2x_3x_4), e_2, e_3),\\ &((x_2x_4x_3), e_2, e_3), (e_1, e_2, (z_2z_3z_4)), (e_1, e_2, (z_2z_4z_3)), (e_1, (y_2y_3y_4), (z_2z_3z_4)),\\ &(e_1, (y_2y_4y_3), (z_2z_4z_3)), ((x_2x_3x_4), e_2, (z_2z_3z_4)), ((x_2x_4x_3), e_2, (z_2z_4z_3)),\\ &((x_2x_3x_4), (y_2y_3y_4), e_3), ((x_2x_4x_3), (y_2y_4y_3), e_3), (x_2x_4x_3), (y_2y_4y_3), (z_2z_3z_4)),\\ &((x_2x_3x_4), (y_2y_3y_4), (z_2z_4z_3))\}\end{aligned}$$

The number of elements in $X \times Y \times Z$ fixed by the element in the stabilizer are given in the Table 3.1.

Table 3.1 Elements of the stabilizer and number of fixed points

Type of ordered triple of permutations	No. of ordered triples of permutations	$\lvert Fix(x_1, y_1, z_1)\rvert$
(e_1, e_2, e_3)	1	64
$(e_1, e_2, (z_2z_3z_4))$	2	16
$(e_1, (y_2y_4y_3), e_3)$	2	16
$((x_2x_3x_4), e_2, e_3)$	2	16
$(e_1, (y_2y_3y_4), (z_2z_3z_4))$	4	4
$((x_2x_3x_4), e_2, (z_2z_3z_4))$	4	4
$((x_2x_3x_4), (y_2y_3y_4), e_3)$	4	4
$((x_2x_3x_4), (y_2y_3y_4), (z_2z_4z_3))$	8	1
TOTAL	27	

Hence by Theorem 2.1, the number of orbits of $G_{(x_1,y_1,z_1)}$ acting on K are given by

$$\frac{1}{\left|G_{(x_1,y_1,z_1)}\right|}\sum_{g\in G_{(x_1,y_1,z_1)}}\left|Fix(g)\right| = \frac{1}{27}[1\times 27+2\times 16+2\times 16+2\times 16+4\times 4+4\times 4+4\times 4+8\times 1]$$

$$=\frac{216}{8}=8$$

We now obtain the orbits of the $G_{(x_1,y_1,z_1)}$. The orbit containing a point *x*, where *x* is an ordered triple from $X\times Y\times Z$ is obtained by multiplying the permutations of $G_{(x_1,y_1,z_1)}$ with *x*. The following are orbits of the stabilizer of (x_1,y_1,z_1) which are classified based on how many elements of the set $A=\{x_1,y_1,z_1\}$ they contain;

a) The orbit whose every element contains exactly 3 elements of A,

$$\Delta_0 = Orb_{G_{(x_1,y_1,z_1)}}(x_1,y_1,z_1)=\{(x_1,y_1,z_1)\}, \text{ the trivial suborbit}$$

b) The orbits whose every element contains exactly 2 elements of A,

$$\Delta_1 = Orb_{G_{(x_1,y_1,z_1)}}(x_1,y_{\neq 1},z_1)=\{(x_1,y_2,z_1),(x_1,y_3,z_1),(x_1,y_4,z_1)\}$$

$$\Delta_2 = Orb_{G_{(x_1,y_1,z_1)}}(x_1,y_1,z_{\neq 1})=\{(x_1,y_1,z_2),(x_1,y_1,z_3),(x_1,y_1,z_4)\}$$

$$\Delta_3 = Orb_{G_{(x_1,y_1,z_1)}}(x_{\neq 1},y_1,z_1)=\{(x_2,y_1,z_1),(x_3,y_1,z_1),(x_4,y_1,z_1)\}$$

c) The orbits whose every element contains exactly 1 element of *A*

$$\Delta_4 = Orb_{G_{(x_1,y_1,z_1)}}(x_1,y_{\neq 1},z_{\neq 1})=\{(x_1,y_2,z_2),(x_1,y_2,z_3),(x_1,y_2,z_4),(x_1,y_3,z_2),$$
$$(x_1,y_3,z_3),(x_1,y_3,z_4),(x_1,y_4,z_2),(x_1,y_4,z_3),(x_1,y_4,z_4)\}$$

$$\Delta_5 = Orb_{G_{(x_1,y_1,z_1)}}(x_{\neq 1},y_1,z_{\neq 1})=\{(x_2,y_1,z_2),(x_2,y_1,z_3),(x_2,y_1,z_4),(x_3,y_1,z_2),$$
$$(x_3,y_1,z_3),(x_3,y_1,z_4),(x_4,y_1,z_2),(x_4,y_1,z_3),(x_4,y_1,z_4)\}$$

$$\Delta_6 = Orb_{G_{(x_1,y_1,z_1)}}(x_{\neq 1},y_{\neq 1},z_1)=\{(x_2,y_2,z_1),(x_2,y_3,z_1),(x_2,y_4,z_4),(x_3,y_2,z_1),$$
$$(x_3,y_3,z_1),(x_3,y_4,z_1),(x_4,y_2,z_1),(x_4,y_3,z_1),(x_4,y_4,z_1)\}$$

d) The orbits whose every element contains no element of *A*

$$\Delta_7 = Orb_{G_{(x_1,y_1,z_1)}}(x_{\neq 1},y_{\neq 1},z_{\neq 1})=\{(x_2,y_2,z_2),(x_2,y_2,z_3),(x_2,y_2,z_4),(x_2,y_3,z_2),(x_2,y_3,z_3),(x_2,y_3,z_4),$$
$$(x_2,y_4,z_2),(x_2,y_4,z_3),(x_2,y_4,z_4),(x_3,y_2,z_2),(x_3,y_2,z_3),(x_3,y_2,z_4),(x_3,y_3,z_2),(x_3,y_3,z_3),(x_3,y_3,z_4),$$
$$(x_3,y_4,z_2),(x_3,y_4,z_3),(x_3,y_4,z_4),(x_4,y_2,z_2),(x_4,y_2,z_3),(x_4,y_2,z_4),(x_4,y_3,z_2),(x_4,y_3,z_3),(x_4,y_3,z_4),$$
$$(x_4,y_4,z_2),(x_4,y_4,z_3),(x_4,y_4,z_4)\}$$

The sub degrees are summarized in the Table 3.2

Table 3.2 Orbits of the Stabilizer and Sub degrees

Types Orbits	Subdegrees
Trivial orbit	1
Orbit containing exactly 2 element of *A*	3
Orbit containing exactly 1 element of *A*	9
Orbit containing exactly no element of *A*	27

Lemma 3.4. If (A_5, X), (A_5, Y) and (A_5, Z) are alternating groups with $X = \{x_1, x_2, x_3, x_4, x_5\}, Y = \{y_1, y_2, y_3, y_4, y_5\}$ and $Z = \{z_1, z_2, z_3, z_4, z_5\}$, then the rank of $A_5 \times A_5 \times A_5$ acting on $X \times Y \times Z$ is 8 and the subdegrees are 1, 4, 25, 64.

Proof. Let $G = A_5 \times A_5 \times A_5$ and $K = X \times Y \times Z$. If (x_1, y_1, z_1) represents an arbitrary element from *K*, then $stab_G(x_1, y_1, z_1)$ and the number of fixed points are as in the Table 3.3.

Table 3.3 Elements of the stabilizer and number of fixed points

Type of ordered triple of permutations	No. of ordered triples of permutations	$\lvert Fix(x_1, y_1, z_1)\rvert$	Total number of fixed points
(e_1, e_2, e_3)	1	125	125
$(e_1, e_2, (abc))$	8	50	400
$(e_1, e_2, (ab)(cd))$	3	25	75
$(e_1, (abc), e_3)$	8	50	400
$(e_1, (ab)(cd), e_3)$	3	25	75
$((abc), e_2, e_3)$	8	50	400

$((ab)(cd),e_2,e_3)$	3	25	75
$(e_1,(abc),(def))$	64	20	1280
$(e_1,(abc),(de)(fg))$	24	10	240
$(e_1,(ab)(cd),(efg))$	24	10	240
$(e_1,(ab)(cd),(ef)(gh))$	9	5	45
$((abc),e_2,(def))$	64	20	1280
$((abc),e_2,(de)(fg))$	24	10	240
$((ab)(cd),e_2,(efg))$	24	10	240
$((ab)(cd),e_2,(ef)(gh))$	9	5	45
$((abc),(def),e_3)$	64	20	1280
$((abc),(de)(fg),e_3)$	24	10	240
$((ab)(cd),(efg),e_3)$	24	10	240
$((ab)(cd),(ef)(gh),e_3)$	9	5	45
$((abc),(def),(ghi))$	512	8	4096
$((abc),(def),(gh)(ij))$	192	4	768
$((abc),(de)(fg),(hij))$	192	4	768
$((ab)(cd),(efg),(hij))$	192	4	768
$((abc),(de)(fg),(hi)(jk))$	72	2	144
$((ab)(cd),(efg),(hi)(jk))$	72	2	144
$((ab)(cd),(ef)(gh),(ijk))$	72	2	144
$((ab)(cd),(ef)(gh),(ij)(kl))$	27	1	27
TOTAL	**1728**		**13824**

Hence by Theorem 2.1, the number of orbits of $G_{(x_1,y_1,z_1)}$ acting on K are given by

$$\frac{1}{|G_{(x_1,y_1,z_1)}|}\sum_{g\in G_{(x_1,y_1,z_1)}}|Fix(g)| = \frac{13824}{1728}$$
$$= 8$$

The orbits of the stabilizer of (x_1, y_1, z_1) which are classified based on how many elements of the set $A = \{x_1, y_1, z_1\}$ they contain are shown in Table 3.4.

Table 3.4 Orbits of the Stabilizer and their length

Type of orbit	Number of orbits	Subdegrees
The orbit containing exactly 3 elements of *A*	1	1
The orbits containing exactly 2 elements of *A*	3	4
The orbits containing exactly 1 element of *A*	3	25
The orbits containing no element of *A*	1	64

Theorem 3.5. If (A_n, X), (A_n, Y) and (A_n, Z) are alternating groups with $X = \{x_1, x_2, ..., x_n\}, Y = \{y_1, y_2, ..., y_n\}$ and $Z = \{z_1, z_2, ..., z_n\}$, and if $n \geq 4$, then the rank of $A_n \times A_n \times A_n$ acting on $X \times Y \times Z$ is 8 and the subdegrees are 1, $(n-1), (n-1)^2, (n-1)^3$.

Proof. Let $G = A_n \times A_n \times A_n$ and $K = X \times Y \times Z$, then $stab_G(x_1, y_1, z_1)$ is isomorphic to $A_{n-1} \times A_{n-1} \times A_{n-1}$ where the permuting elements are from the sets $X - \{x_1\}, Y - \{y_1\}$ and $Z - \{z_1\}$.

There are 8 orbits of the stabilizer of the point set (x_1, y_1, z_1) which are classified based on how many elements of the set $A = \{x_1, y_1, z_1\}$ they contain. The orbits and their lengths listed on the Table 3.5.

Table 3.5 Orbits of $stab_G(x_1, y_1, z_1)$ **and the subdegrees**

Type of orbit	Number of orbits	Subdegrees
The orbit containing exactly 3 elements of *A*	1	1
The orbits containing exactly 2 elements of *A*	3	$n-1$
The orbits containing exactly 1 element of *A*	3	$(n-1)^2$
The orbits containing no element of *A*	1	$(n-1)^3$

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