Polynomials of Unbounded Positive Self-adjoint Operators

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Abstract

The polynomials of unbounded Self-adjoint operators are not necessarily Self-adjoint because as much as their terms may be densely defined, the sum of the terms may not be densely defined. In this paper, we provide the conditions under which the polynomials may be Self-adjoint. This is an extension of the results by Mortad on the sum of two densely defined unbounded Self-adjoint operators. We achieve this by limiting the choice of our operators to invertible unbounded self-adjoint operators in which strictly positive unbounded Self-adjoint operators are part of.

Keywords: Unbounded operators, polynomials of operators, Self-adjoint operators polynomials, strictly positive operators

1. Introduction

The Unbounded Self-adjoint operators are classified as densely defined closed linear operators. This implies that the denseness of the domain of these operators is fundamental in defining the Unbounded Self-adjoint operators. The concept of denseness makes it impossible to assume that the sum of any two densely defined operators is densely defined. Consequently, the sum of unbounded Self-adjoint operators is not necessarily an unbounded Self-adjoint operator. This fact, further, implies that the polynomials of the unbounded Self-adjoint operator are not generally unbounded Self-adjoint operators. This is the problem that we would like to investigate and provide conditions under which it could be true.

When E and F are bounded Self-adjoint operators, then the sum $E + F$ is always Self-adjoint since

$$(E + F)^* = E^* + F^* = E + F.$$

However, if the operators turn out to be unbounded Self-adjoint operators, then we have

$$(E + F)^* \supseteq E^* + F^* = E + F. \qquad (1.1)$$

The relation in equation 1.1 makes it difficult for one to define an unbounded Self-adjoint operator $p_n(F)$ such that

$$p_n(F) = \sum_{k=0}^{n} a_k F^k \quad (1.2)$$

where$F^k, k = 1,2, \dots, n$ are unbounded Self-adjoint operators and $a_k$are constants.

From the general properties of polynomials of densely defined unbounded operators, we have that domainis given by $D\big(p_n(F)\big) = D(F^n)$ where n is the degree of the polynomial$p_n(F)$. When the operators F^k are bounded, then 1.2 is a bounded Self-adjoint operator as shown in **[4]**. Our interest to establish the polynomial in equation 1.2 where F^k 's are unbounded.

In this paper, we will denote a complex Hilbert space by $\mathbb{H}$ and a real number set by $\mathbb{R}$. We will mostly use F to refer to our unbounded Self-adjoint operator where its power will be denoted by $F^k, k = 1,2, \dots, n$. The powers of F are defined as operator composition $F^k = F(F^{k-1})$.

Definition: Positive operator

An operator F is positive if $\langle Fx, x\rangle \geq 0$ and *strictly positive* if $\langle Fx, x\rangle > 0$ for $x \in D(F)$.

Definition: Spectral measure

If $\mathfrak{X}$ is a σ-algebra of a non-empty set X, then spectral measure is a function $P(\cdot): \mathfrak{X} \to \mathbb{H}$ where

1) $P^2(A) = P(A)$ and $P^*(A) = P(A)$ for $A \in \mathfrak{X}$
2) $P(X) = \mathbf{I}$
3) $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ for $A_i \in \mathfrak{X},\ i \in I :\ A_i \cap A_j = \emptyset$ where $i \neq j$ and $\bigcup_{i=1}^{\infty} A_i = X$.

Definition: Essential range

If $(X, \mathfrak{X}, \mu)$ is a measure space and f a Borel-measurable function. Then the essential range of f is

$$\{z \in \mathbb{C}: \mu(\{u \in X: |f(u) - z| < \varepsilon\}) > 0 \; for\ all\ \varepsilon > 0\}$$

We also note that the spectrum of the integral operator $\mathcal{T}(f)$ is its essential range, **[5]**.

2. Preliminary Concepts

The main results of this section area result of the extension of the following theorem by Mortad. The theorem provides a condition under which the adjoints of the sum of two unbounded Self-adjoint operators can be equal to the sum of adjoints of the two operators.

Proposition 2.1

Let E and F be unbounded Self-adjoint operators such that F is invertible and $EF = FE$.

If $D(EF^{-1}) \subseteq D(F)$, then $(E + F)^* = E + F$ is Self-adjoint on $D(E)$.**[3]**

Proof

The adjoints of densely defined operators are always extensions of the corresponding operator, as such $(E + F) \subseteq (E + F)^*$

We now need to establish that $(E + F)^* \subseteq (E + F)$.

The operator $F^{-1}$is an invertible operator hence, bounded **[5]**, as such $F^{-1}E \subseteq EF^{-1}$. Consequently, $D(F^{-1}E) \subseteq D(EF^{-1})$. The domain of the sum of unbounded operators $E + F$ is defined as $D(E + F) = D(E) \cap D(F)$. On the other hand, the domain of the product of two unbounded is defined as $D(F^{-1}E) = D(E)$ given that $Ran(E) \subseteq D(F^{-1})$. This is true since F is assumed to be invertible, hence bijective. Thus, using the assumption $D(EF^{-1}) \subseteq D(F)$, we have

$$D(E) = D(F^{-1}E) \subseteq D(EF^{-1}) \subseteq D(F)$$

Now that $D(E) \subseteq D(F)$, we have $D(E + F) = D(E) \cap D(F) = D(E)$.

We also have that $F^{-1}F \subseteq I$ see**[2],** hence $D(F^{-1}FE) \subset D(E)$, consequently $D(F^{-1}FE + F) \subset D(E + F)$. This implies that $F^{-1}FE + F \subset E + F$. By commutativity of$E$ and F, we have

$$F^{-1}FE + F \subset E + F$$

$$F^{-1}EF + F \subset E + F$$

$$(F^{-1}E + I)F \subset E + F$$

We make use of the following results for adjoints of the unbounded operator, that is$E^* + F^* = (E + F)^*$ if F and E are densely defined unbounded operators where$F$ is bounded. We also have that$(EF)^* = F^*E^*$ when E and F be densely defined operators where F^{-1} exists in $B(\mathbb{H})$, **[3]**. Taking adjoints, we have $(E + F)^* \subseteq [(F^{-1}E + I)F]^*$. The operator F is invertible so is $(F^{-1}E + I$, thus

$$\begin{aligned}(E + F)^* &\subseteq [(F^{-1}E + I)F]^* \\ &= F^*(F^{-1}E + I)^* \\ &= F^*((F^{-1}E)^* + I) \\ &= F^*(E^*(F^{-1})^* + I).\end{aligned}$$

Using the Self-adjointness property of F and E, the adjoints of the inverse of Unbounded Self-adjoint operators with the assumption that $D\big(E(F^{-1})\big) \subseteq D(F)$ and the commutativity of E and F, we get

$$\begin{aligned}(E + F)^* &\subseteq F^*(E^*(F^{-1})^* + I) \\ &= F(EF^{-1} + I) \\ &= FEF^{-1} + F \\ &= EFF^{-1} + F \\ &= E + F.\end{aligned}$$

The two inclusions above imply that $(E + F)^* = E + F$.□

To establish the invertibility of the polynomials of the operators, the spectral mapping theorem will play an important role. We therefore state and prove it.

Proposition 2.2 Spectral mapping theorem

Let F be an unbounded operator and $g: \sigma(F) \to \mathbb{C}$ be a continuous function, then

$$\sigma(g(F)) = \overline{g(\sigma(F))}, \text{ [5]}$$

Proof

First, we provethat $\overline{g(\sigma(F))} \subseteq \sigma(g(F))$.

Let $z_0 \in \overline{g(\sigma(F))}$, then for $\varepsilon > 0$, there exists $\lambda_0 \in \sigma(F)$ such that $|z_0 - g(\lambda_0)| < \frac{\varepsilon}{2}$. Since g is continuous, there is a $\delta > 0$ such that

$$\{\lambda: |\lambda - \lambda_0| < \delta\} \subseteq \left\{\lambda: |g(\lambda) - g(\lambda_0)| < \frac{\varepsilon}{2}\right\} \subseteq \{\lambda: |f(\lambda) - z_0| < \varepsilon\}$$

By choice, $\lambda_0 \in \sigma(F)$. Since $\sigma(F) = \text{support of} P_F(\cdot)$, a closed compact set, we have $P_F(\{ \lambda: |\lambda - \lambda_0| < \delta \}) > 0$. Consequently, $P_F(\{\lambda : |f(\lambda) - z_0|\}) > 0$ implying that z_0 is in the essential range of g. By definition, the essential range of g we have that $z_0 \in \sigma(g(F))$.

Conversely, let $z_0 \notin \overline{g(\sigma(F))}$ then $\{\lambda : | f(\lambda) - z_0|\} = \emptyset$ implying that $P_F(\{\lambda : |f(\lambda) - z_0|\}) = 0$ for $\varepsilon > 0$. This implies that z_0 is not in the essential range of g, consequently, $z_0 \notin \sigma(g(F))$, a contrapositive of $\sigma(g(F)) \subseteq \overline{g(\sigma(F))}$.□

The closed densely defined operator has a closed spectrum, **[1]**. We also have that Unbounded Self-adjoint operators are closed and its spectrum is a subset of the real number set, it is closed and bounded, hence compact. If F is an unbounded Self-adjoint operator then from the spectral mapping theorem we have

$$\sigma(g(F)) = \overline{g(\sigma(F))} = g(\sigma(F)). \qquad (2.1)$$

3. Inverses of powers of Linear Operators

In this section, we provide the definition of powers of linear operators as well as the condition for them to be invertible.

The n-the power of a linear operator, for this cases, Unbounded Self-adjoint operator,are given defined recursively as $F^n = F(F^{n-1})$ for $n \in \mathbb{N}$ and $F^0 = I$. The operator F^n is Unbounded Self-adjoint operator since F is, however, its domain is suitably chosen so that it is dense in $\mathbb{H}$. The domain of F becomes smaller than that of F dues to subsequent application of F to the lower power, that is $D(F^n) \subseteq D(F)$.

The invertibility of the powers of F depends on that of F. If $Ker(F_1) = \{0\}$ then F^{-1} exists. Furthermore, if $Ker(F_1) = \{0\}$ and $Ker(F_2) = \{0\}$ then $Ker(F_2F_1) = \{0\}$, $(F_1F_2)^{-1}$ exists and is equal to $F_2^{-1}F_1^{-1}$, **[1]**. With this fact, we move forward to define the inverses of powers of operators.

Lemma 3.1

The power $F^n, n \in \mathbb{N}$ of Unbounded Self-adjoint operator is invertible if F is invertible.

Proof

Consider an operator F with $\lambda_\rho \in \rho(F)$. Then its resolvent is $(F - \lambda_\rho)^{-1}$ implying that $F - \lambda_\rho$ is invertible. If $\lambda_\rho = 0 \in \rho(F)$, then $(F)^{-1} = (F - 0)^{-1}$ exists. Since $\sigma(F) \cap \rho(F) = \emptyset$, we have $0 \in \rho(F)$ and $0 \notin \sigma(F)$. Therefore, F is invertible if and only if $0 \notin \sigma(F)$.

The power function is continuous, therefore, by the spectral mapping theorem, for unbounded Self-adjoint operator

$$0 \notin g(\sigma(F)) = \sigma(g(F)).$$

since $0 \notin \sigma(F)$. Specifically, $0 \notin \sigma(F^n)$ implying that F^n is invertible if F is invertible.□

Using the results in Proposition 2.1 and the condition for invertibility of powers of Unbounded Self-adjoint Operators, we can form Polynomials of these Operators.

4. Polynomials of Unbounded Self-adjoint Operators

In this section, we form the polynomials of Unbounded Self-adjoint operators and provide conditions for their existence. In this section, we will refer to Polynomials of Unbounded Self-adjoint operators as simply polynomial of operators. We will treat a term in a polynomial as an Unbounded Self-adjoint operator for the direct application of proposition 2.1. However, this will not go down without imposing some conditions on the coefficients of the terms in the polynomial.

Lemma 4.1

The coefficients of powers of the Self-adjoint operator in a Self-adjoint Polynomial are real.

Proof

We will define our polynomial as

$$p_n(F) = \sum_{k=0}^{n} a_k F^k,\ n \in \mathbb{N}. \qquad (4.1)$$

Consider a general term $a_k F^k$ of the polynomial. Since it must be Self-adjoint, we must have

$$(a_k F^k)^* = \overline{a_k (F^k)^*} = \overline{a_k (F^*)^k} = a_k F^k$$

Thus, we must have $\overline{a_k} = a_k$ which is only possible when $a_k \in \mathbb{R}$. This implies that our coefficients will all be real. □

Lemma 4.2

Two unbounded Self-adjoint operators commute.

Proof

The adjoint operators are extensions of their respective operators **[1]** implying that

$$E_1E_2 \subseteq (E_1E_2)^* \supseteq E_2^*E_1^* = E_2E_1 \quad (4.2)$$

Thus implies that $E_1E_2 = E_2E_1$.□

We now proceed to form polynomials of Self-adjoint operators from degree 1.

The Self-adjoint operator polynomials of degree 0 is simply a real number, hence, we begin with degree 1.

Illustration 1: Polynomials of degree 1

A degree 1 polynomial will generally be given by

$$p_1(F) = a_0I + a_1F \quad (4.3)$$

The operator F is invertible, by choice. By Spectral mapping theorem $0 \notin \sigma(a_1F)$ since $0 \notin \sigma(F)$. Therefore, a_1F is invertible.

For the case of commutativity, $a_0I \in \mathbb{R}$ commutes with any Self-adjoint operator hence the two terms are commutative.

Finally, we have that if $D(IF^{-1}) = D(F^{-1}) \subseteq D(F)$, then the hypothesis of proposition 2.1 are satisfied, hence, we have that $p_1(F)$ is Self-adjoint.

Illustration 2: Polynomials of degree

A degree 2 polynomial will generally be given by

$$p_2(F) = (a_0I + a_1F) + a_2F^2 \quad (4.4)$$

The operator F^2 is invertible, by lemma 3.1. Using the Spectral mapping theorem a_2F^2 is invertible as well.

For the case of commutativity, if $E_1 = p_1(F)$ and $E_2 = a_2F^2$, we get two commutative operators by lemma 4.2.

Finally, we have that if $D(F(F^2)^{-1}) \subseteq D(F^2)$, then the hypothesis of proposition 2.1 are satisfied, hence, we have that $p_2(F)$ is Self-adjoint.

We now generalize of a polynomial $p_n(F)$ of degree n.

Proposition 4.1

The polynomial of Unbounded Self-adjoint operators, $p_n(F) = \sum_{k=0}^{n} a_k F^k$, $n \in \mathbb{N}$, is an unbounded Self-adjoint operator on $D(F^{n-1})$ if $D(F^{n-1}(F^n)^{-1}) \subseteq D(F^n)$.

Proof

We prove this theorem by mathematical induction

When we have a polynomial of degree 1, the operator $p_1(F)$ is Self-adjoint as proved in Illustration 1.

Assume that $p_m(F)$ is true for some $m \in \mathbb{N}$, then the polynomial is Self-adjoint. We then prove that $p_{m+1}(F)$ is Self-adjoint.

The operator F^{m+1} is invertible, by lemma 3.1. Using the Spectral mapping theorem $a_{m+1}F^{m+1}$ is invertible as well.

For the case of commutativity, if $E_1 = p_m(F)$ and $E_2 = a_{m+1}F^{m+1}$, we get two commutative operators by lemma 4.2.

Finally, we have that if $D(F^m(F^{m+1})^{-1}) \subseteq D(F^{m+1})$, then the hypothesis of proposition 2.1 are satisfied, hence, we have that $p_{m+1}(F)$ is Self-adjoint on.

Since $p_1(F)$, $p_m(F)$ and $p_{m+1}(F)$are Self-adjoint, we have that $p_n(F)$ is Unbounded Self-adjoint Operator for all $n \in \mathbb{N}$ on the domain $D(F^{n-1})$.□

The above proposition works for unbounded strictly positive Self-adjoint operators since their spectrum does not contain zero implying that they are invertible.

Corollary4.1

The polynomial of Unbounded strictly positive Self-adjoint operators, $p_n(F) = \sum_{k=0}^{n} a_k F^k$, $n \in \mathbb{N}$, is an unbounded positive Self-adjoint operator if $D(F^{n-1}(F^n)^{-1}) \subseteq D(F^n)$.

Proof

Let F be a strictly positive Self-adjoint operator, then $\sigma(F) \in (0,\infty)$, [5]. By lemmas 3.1 and 4.1, F and all its integers powers are Self-adjoint and invertible. By lemma 4.1, 4.2, and proposition 4.1 we can form polynomial of Self-adjoint operators in F if $D(F^{n-1}(F^n)^{-1}) \subseteq D(F^n)$.□

Having come up with polynomials of unbounded Self-adjoint operators under the stated conditions, we state its spectral decomposition.

If F is an unbounded Self-adjoint operator then by the spectral theorem for unbounded Self-adjoint operator, there exists a unique spectral measure $P_F(\cdot)$ such that

$$F = \int_{\mathbb{R}} t dP_F(t), \quad t \in \sigma(F) \tag{4.5}$$

Furthermore, if g is a P_F-a.e finite Borel function then

$$g(F) = \int_{\mathbb{R}} f(t) dP_F(t),\ t \in \sigma(F).\ [\mathbf{5}] \tag{4.6}$$

In our case, F is an unbounded Self-adjoint operator, hence by the spectral theorem, there is a unique spectral measure $P\_F(\cdot)$ such that F is given by equation 4.5 above.

A polynomial is continuous, as such, it is P_F-a.e finite Borel function, thus, by equation 4.6, we have the spectral decomposition

$$p_n(F) = \int_{\mathbb{R}} p_n(t) dP_F(t), \qquad t \in \sigma(F). \tag{4.7}$$

where $P_F(\cdot)$ is the spectral measure of the operator F.

Therefore, the spectral measure for our polynomial, $p_n(F) = \sum_{k=0}^{n} a_k F^k$, $n \in \mathbb{N}$, in proposition 4.1 is the spectral measure of the operator F, that is, $P_F(\cdot)$.

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