GENERALIZED SUBORBITAL GRAPHS OF THE DIHEDRAL GROUP D_n ACTING ON UNORDERED SUBSETS

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Abstract
The study examines some properties of suborbital graphs of the dihedral group D_n acting on $X^{(r)}$, the set of all unordered r-element subsets of X={1, 2, …, n}. It has been shown that the graph Γ_i is undirected for all i, connected only if the pair (i, n) is coprime, and has girth n/d where d is the gcd (i, n). Equivalence of transitive actions has been proved and the generalized graphs constructed.
Key words: Transitive action, Suborbit, Suborbital graph, Girth, Connected component.

1. Introduction

Suborbital graphs associated with various group actions have been examined (See (Hamma & Aliyu, 2010; Kamuti *et al*, 2012; Rimberia *et al*, 2012; Keskin, 2006; Akbas & Baskan, 1996; Jones *et al*, 1991; Guler *et al*, 2008; Sims, 1967; Olum, 2019; Gachogu *et al*, 2019)).
In this paper, the generalized suborbital graphs associated with the action of D_n on $X^{(r)}$ have been constructed and their properties discussed.

2. Preliminary Definitions

Definition 2.1 (Transitive action)
Let G act on a set X and $x\in X$. The orbit of x is the set, $orb_G(x)=\{gx\in X \mid g\in G\}$. If the action has only one orbit, then G is said to act transitively on X.

Definition 2.2 (Suborbit)

Let $G=D_n$ act on a set X. The stabilizer of a point $x \in X$ is the set of all elements $g \in G$ which fix x, denoted by $G_x=\{ g \in G \mid gx=x \}$. The orbits, $\Delta_i=\{i+1\}$ (i=0, 1, …, n-1), of G_x on X are known as suborbits of G.

Definition 2.3 (Paired suborbits)

Let G act transitively on a set X and let Δ be an orbit of G_x on X. Define $\Delta^*=\{gx \mid g\in G, x\in g\Delta\}$. Then Δ^* is also an orbit of G_x and is called the G_x-orbit paired with Δ. If $\Delta=\Delta^*$, then Δ is said to be self-paired.

Theorem 2.1

Let G act transitively on a set X, and suppose $g \in G$. The number of self-paired suborbits of G is given by $\frac{1}{|G|}\sum_{g\in G}|Fix(g^2)|$ (Cameron, 1994).

Definition 2.4 (Suborbitals)

Suppose G is a transitive group acting on X. The action of G on $X\times X$ is defined by; $g(x, y)=(gx, gy)$, $g \in G$, x, $y \in X$. The orbits of this action are known as suborbitals of G. The suborbital of G on X corresponding to Δ_i is the set $O_i=\{g(\Delta_0, A_i) \mid g \in G, A_i \in\Delta_i \}$.

Definition 2.5 (Suborbital graph)

The suborbital graph Γ_i corresponding to the suborbital O_i is formed by taking elements of X as the set of vertices of Γ_i and by drawing a directed edge from x to y if and only if $(x, y) \in O_i=\{g(\Delta_0\ A_i)|$ $g\in G, A_i\in\Delta_i\}$. The graph is undirected if and only if for all $(x, y) \in O_i$, $(y, x) \in O_i$. Equivalently, Γ_i is undirected only if Δ_i is self-paired. Hence each suborbital O_i determines a suborbital graph Γ_i, which corresponds to Δ_i.

Definition 2.6 (Equivalent actions)

Let (G_1, S_1) and (G_2, S_2) be permutation groups, where G_i acts on S_i. The permutation isomorphism, $(G_1, S_1)\cong(G_2, S_2)$, means that there exists a group isomorphism $\phi: G_1\rightarrow G_2$ and a bijection $\Theta: S_1\rightarrow S_2$ so that $\Theta(gs)=\phi(g)\Theta(s)$ for all $g \in G_1$, $s \in S_1$.

Theorem 2.2 (Orbit-Stabilizer Theorem)

Let G be a group acting on a finite set X with x in X. The size of the orbit of x in G is the index $|G : G_x|$. Thus, $|orb_G(x)|=|G : G_x|$ (Rose, 1978)

3. Main Results

Theorem 3.1

$G= D_n$ acts transitively on $X^{(r)}$ if and only if $r=1$, $r=n-1$ or $r=n$.

Proof:

Let D_n act on X and $x\in X$. The size $|G_x|=2$, since $g\in G_x$ if and only if g is the identity or a reflection along the line passing through vertex x. From Theorem 2.2, $|orb_G(x)|=2n/2=n$. Now, $n=|X^{(r)}|=\binom{n}{r}$, $\Rightarrow r=1$ or $r=n-1$. Conversely, if $r=1$ or $r=n-1$, then $|X^{(r)}|=\binom{n}{r}=|orb_G(x)|$. Clearly, every $g \in G$ fixes 1 element in $X^{(n)}$. It follows, $|orb_G(x)|=2n/2n=1=|X^{(n)}|$. But this is a trivial action on 1 element.

Theorem 3.2

The action of D_n on X has exactly1 suborbit of length 1 and (n-1)/2 suborbits of length 2 when n is odd. But there are 2 suborbits of length 1 and (n-2)/2 suborbits of length 2 when n is even.

Proof:

When n is odd, $G_1=\{1, (2\ n)(3\ n\text{-}1)\ldots(\frac{n+1}{2}\ \frac{n+3}{2})\}$. Now, $g\in G_1$ is such that $g(x) = x$ or $g(x) = n+2-x$ for all $x \in X$. The elements in a suborbit of length 1 are given by solving $x=n+2-x \mod n$. It follows, x=1 when n is odd, Δ_0. But, x=1 or (n+2)/2 *mod n* when n is even and $G_1=\{1, (2\ n)(3\ n\text{-}1)\ldots\left(\frac{n+2}{2}\right)\}$. The number of suborbits of length 2 is the number of pairs from the remaining (n-1) and (n-2) elements respectively, (n-1)/2 when n is odd and (n-2)/2 when n is even.

Theorem 3.3

The suborbits of D_n on X are of the form;

$\Delta_i=\{i+1, n+1-i\}$, where i=0, 1, …, $\frac{n-1}{2}$, when n is odd, and i=0, 1, …, $\frac{n}{2}$, when n is even.

Proof:

Let Δ_i be the the G_1-orbit of i+1. By Theorem 3.2, $\Delta_i=\{g(i+1)|g\in G_1\}=i+1$ or $n+2-(i+1)$. It follows, $\Delta_i=\{i+1, n+1-i\}$, where i=0, 1, …, $\frac{n-1}{2}$, when n is odd, and i=0, 1, …, $\frac{n}{2}$, when n is even. The suborbits are then as follows;

$\Delta_0=\{1\}$, $\Delta_1=\{2, n\}$, $\Delta_2=\{3, n-1\}$, …, $\Delta_{\frac{n-1}{2}}=\left\{\frac{n+1}{2}, \frac{n+3}{2}\right\}$, when n is odd.

$\Delta_0=\{1\}$, $\Delta_1=\{2, n\}$, $\Delta_2=\{3, n-1\}$, …, $\Delta_{\frac{n}{2}}=\{\frac{n+2}{2}\}$, when n is even.

Theorem 3.4

The number of suborbits of D_n on X is (n+1)/2 when n is odd and (n+2)/2 when n is even.

Proof:

The proof follows from Theorem 3.2, where the total number of suborbits is ((n-1)/2 +1)=(n+1)/2, when n is odd and ((n-2)/2 +2)=(n+2)/2 when n is even.

Example 3.1

Let D_{10} act on X. The six suborbits are;

$\Delta_0=\{1\}$, $\Delta_1=\{2, 10\}$, $\Delta_2=\{3, 9\}$, $\Delta_3=\{4, 8\}$, $\Delta_4=\{5, 7\}$, $\Delta_5=\{6\}$.

Theorem 3.5

The number of self-paired suborbits of D_n on X is(n+1)/2 when n is odd and (n+2)/2 when n is even.

Proof:

Let D_n act on X and $x \in X$. When n is odd, $g^2 \in D_n$ fixes x if g is the identity or a reflection along a line through vertex x. From the identity, g^2 fixes n elements in X. From each reflection, g^2 fixes n elements. The number of elements fixed by n reflections is n^2. By Theorem 2.1, the number of self-paired suborbits is $(n^2+n)/2n=(n+1)/2$. When n is even, the proof is similar, with an additional g^2 from a rotation of 180^0. The rotation g^2 fixes n elements and the number of self-paired suborbits is, $(n^2+2n)/2n=(n+2)/2$. □

Corollary 3.1

All suborbits of D_n on X are self-paired.

Proof:
From Theorems 3.4 and 3.5, the number of self-paired suborbits equals the number of suborbits. Hence, the proof.

Theorem 3.6

The number of suborbits of D_n on $X^{(n-1)}$ is $(n+1)/2$ when n is odd and $(n+2)/2$ when n is even.

Proof:

Let G_1 act on X. For every $g \in G_1$, $g(1)=1$, $\Rightarrow g\{2, 3, \ldots, n\}=\{2, 3, \ldots, n\} \in X^{(n-1)}$. Whenever g in G_1 stabilizes 1, it automatically stabilizes $\{2, 3, \ldots, n\} \in X^{(n-1)}$. The action of G_1 on X induces the action of $G_{\{2, 3, \ldots, n\}}$ on $X^{(n-1)}$. Hence, the number of $G_{\{2, 3, \ldots, n\}}$-orbits on $X^{(n-1)}$ equals the number of G_1-orbits on X, $(n+1)/2$ when n is odd and $(n+2)/2$ when n is even, as shown in Theorem 3.4.

Theorem 3.7
The suborbits of D_n on $X^{(n-1)}$ are of the form;
$\Delta_i=\{\{i+1, i+2, \ldots, i-1, \{1-i, 2-i, \ldots, n-1-i\}\}$, where $(i=0, 1, \ldots, \frac{n-1}{2})$ when n is odd, and $(i=0, 1, \ldots, \frac{n}{2})$ when n is even.

Proof:
Let D_n act on X. When n is odd, $G_n=\{1, (1\ n-1)(2\ n-2)\ldots\left(\frac{n-1}{2}\ \frac{n+1}{2}\right)\}=G_{\{1, 2, \ldots, n-1\}}$. Whenever g in G_n stabilizes n, it automatically stabilizes $\{1, 2, \ldots, n-1\} \in X^{(n-1)}$. The $G_{\{1, 2, \ldots, n-1\}}$–orbit of $A \in X^{(n-1)}$ is given by $\{A, \{n-x\} |\ x \in A\}$. If Δ_i is the $G_{\{1, 2, \ldots, n-1\}}$– orbit of $\{i+1, i+2, \ldots, i-1\}$, then
$\Delta_i=\{\{i+1, i+2, \ldots, i-1\}, \{1-i, 2-i, \ldots, n-1-i\}\}$, where $(i=0, 1, \ldots, \frac{n-1}{2})$ when n is odd, and $(i=0, 1, \ldots, \frac{n}{2})$ when n is even.

Theorem 3.8
The number of self-paired suborbits of D_n on $X^{(n-1)}$ is $(n+1)/2$ when n is odd, and $(n+2)/2$ when n is even.

Proof:

Let D_n act on X. When n is odd, $g^2 \in D_n$ fixes $x \in X^{(n-1)}$ if g is the identity or a reflection along a line through vertex x. From the identity, g^2 fixes n elements in $X^{(n-1)}$. From each reflection, g^2 fixes n elements. The number of elements fixed by n reflections is n^2. By Theorem 2.1, the number of self-paired suborbits is $(n^2+n)/2n=(n+1)/2$.

When n is even, g^2 fixes x if g is the identity, or a reflection or a rotation of 180^0. The identity g^2 fixes n elements in $X^{(n-1)}$, the n reflections fix n^2 elements and the rotation g^2 fixes n elements. The number of self-paired suborbits is, $(n^2+2n)/2n=(n+2)/2$. □

Corollary 3.2

All suborbits of D_n on $X^{(n\text{-}1)}$ are self-paired.

Proof:

From Theorem 3.6 and Theorem 3.8, the number of self-paired suborbits equals the number of suborbits. Hence the proof.

Corollary 3.3

All suborbital graphs of D_n on X and those of D_n on $X^{(n\text{-}1)}$ are undirected.

Proof:

From Corollary 3.1 and Corollary 3.2, the respective suborbits are self-paired. It follows, the respective graphs are undirected from Definition 2.5.

Remark 3.1

The suborbital O_i of D_n on X is the set $O_i=\{g(\Delta_0, A_i) \mid g \in G\}$, $A_i=\{i+1\} \in \Delta_i=\{\{i+1\}, \{n+1-i\}\}$.

Theorem 3.9
The pair $(x, y) \in O_i$, if and only if $y\text{-}x=i \ mod \ n$ or $y\text{-}x=n\text{-}i \ mod \ n$.
Proof:
Let $(x, y) \in O_i$. Then $(x, y)=hg^j(\Delta_0, A_i)$, where $h(x, y)=(y, x)$ and $g=(12\ldots n)$. Now, $(x, y)= hg^j$ $(1, i+1)=(1+j, i+1+j)$ or $(i+1+j, 1+j) \Longrightarrow y\text{-}x=i \ mod \ n$ or $n\text{-}i \ mod \ n$. Conversely, if $y\text{-}x=i \ mod \ n$ or $y\text{-}x= n\text{-}i \ mod \ n$, then $g^{(x\text{-}1)}(1, i+1)=(x, y)$. Next, there exists h in D_n such that $h(x, y)=(y, x)$, $\Longrightarrow (x, y) \in O_i$.

Theorem 3.10
The graph Γ_i is connected if and only if i and n are coprime.

Proof:

From Theorem 3.9, the cycles of Γ_i correspond to the cycles of g^i, where $g=(12\ldots n)$. The graph is connected only if g^i consists of 1, n-cycle. From the theory of cyclic groups, this is possible if and only if i and n are coprime.

Theorem 3.11
The girth of Γ_i is n/d, where d is gcd (i, n).

Proof:
From Theorem 3.9, the girth of Γ_i is the smallest positive integer k such that $(g^i)^k=1 \ mod \ n$. If i and n have s common divisors, then $k = n/d$, where d is gcd (i, n).

Theorem 3.12
The number of connected components in Γ_i is d, the gcd of i and n.
Proof:
From Theorem 3.11, there is a path in Γ_i from x to y if and only $x \equiv y \bmod d$. A complete residue system *modulo d* has d congruent classes, which are, 0, 1, 2, …, d-1.

Remark 3.2

The suborbital O_i of D_n on $X^{(n-1)}$ is the set $O_i=\{g(\Delta_0, A_i) \mid g \in G\}$, $A_i=\{i+1, i+2, \ldots, i-1\} \in \Delta_i$. The graph Γ_i corresponding to Δ_i is constructed by considering $X^{(n-1)}$ as the vertices, and drawing an edge from C to D if and only if $(C, D) \in O_i$.

Theorem 3.13

The pair $(C, D)=(\{c_1, c_2, \ldots, c_{n-1}\}, \{d_1, d_2, \ldots, d_{n-1}\}) \in O_i(\Delta_0, A_i)$ if and only if $d_k-c_k=i \bmod n$ or $d_k-c_k=n-i \bmod n$ (k=1, 2, …, n-1).

Proof:

If $(C, D) \in O_i(\Delta_0, A_i)$, then $[c_1, d_1]=hg^j\,[1, i+1] = [1+j, i+1+j]$ or $[i+1+j, 1+j]$, $\Longrightarrow d_1-c_1=i \bmod n$ or $n-i \bmod n$. Since the coordinates of C and D are consecutive integers *mod n*, $d_k-c_k=i \bmod n$ or $d_k-c_k=n-i \bmod n$ (k=1, 2, …, n-1). Conversely, if $d_k-c_k=i \bmod n$, then $d_k=c_k+i$ and $hg^{(ck-k)}\,(k, i+k)=h(c_k, d_k)=(d_k, c_k)$, $\Longrightarrow (C, D) \in O_i$ □

Theorem 3.14
The graph Γ_i of D_n on $X^{(n-1)}$ is connected if and only if i and n are coprime.

Proof:

From Theorem 3.13, Γ_i is a map of the sets A_i corresponding to the permutation g^i, where $g=(12\ldots n)$. The proof follows from Theorem 3.10, and the graph is shown in Figure 3.1below.

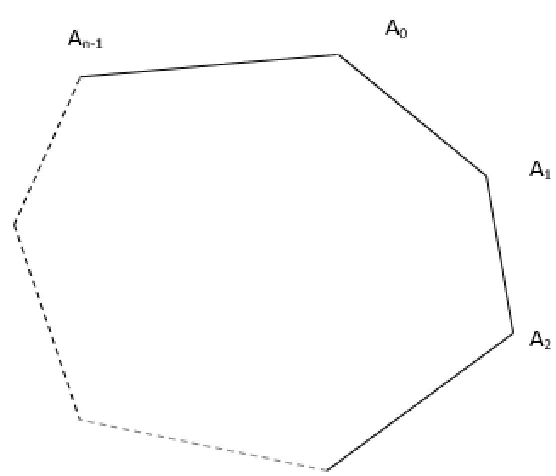

Figure 3.1: The graph Γ_i when i and n are coprime

Theorem 3.15

The graph of Γ_i of D_n on $X^{(n\text{-}1)}$ has d connected components and girth n/d, where d=gcd(i, n).

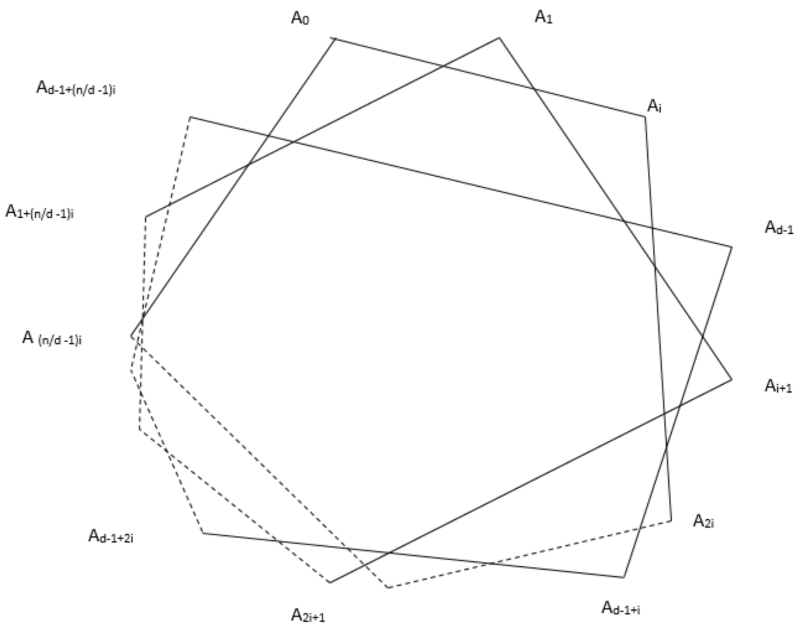

Figure 3.2: The graph Γ_i when gcd (i, n)=d, $d > 1$

Proof:

From Theorems 3.13 and 3.9, the cycles of Γ_i of D_n on $X^{(n\text{-}1)}$ are the cycles of Γ_i of D_n on X. The proof follows from Theorems 3.11 and 3.12. The graph is of the form shown in Figure 3.2.

Corollary 3.4

The graph Γ_i is a star when i=n/2 as shown in Figure 3.3 below.

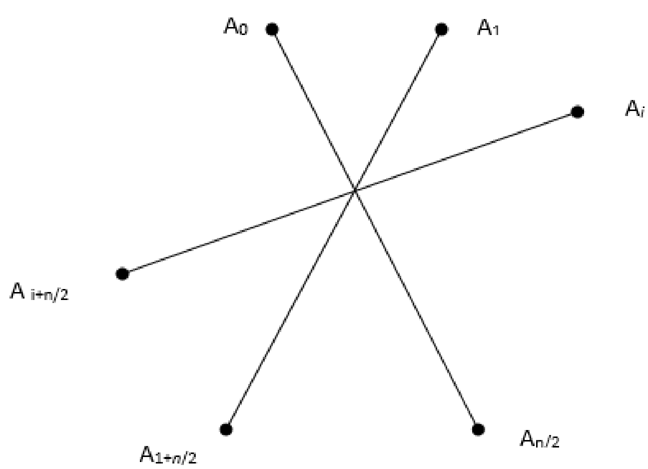

Figure 3.3: The graph Γ_i when i=n/2

Proof:

From Theorem 3.15, when $i=n/2$, $d=n/2$. It follows, Γ_i has cycles of the form, $(\Delta_0\ \Delta_{n/2})(\Delta_1\ \Delta_{1+n/2})\ldots(\Delta_{n-1}\ \Delta_{n/2-1})$. The graph appears as shown in Figure 3.3.

Example 3.2

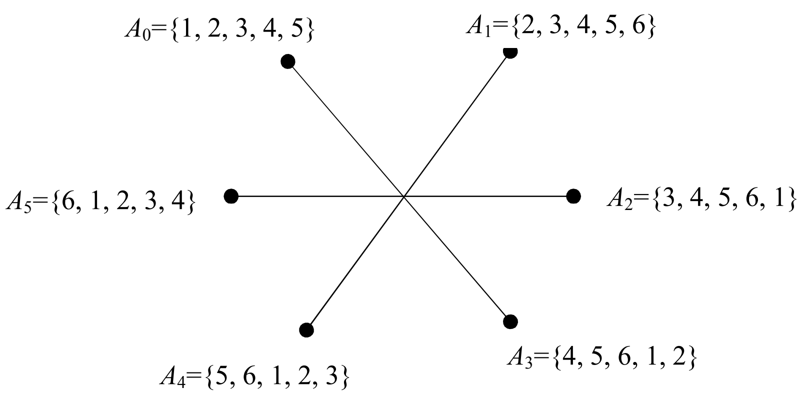

Figure 3.4 *a:* The graph Γ_3 of D_6 on $X^{(5)}$

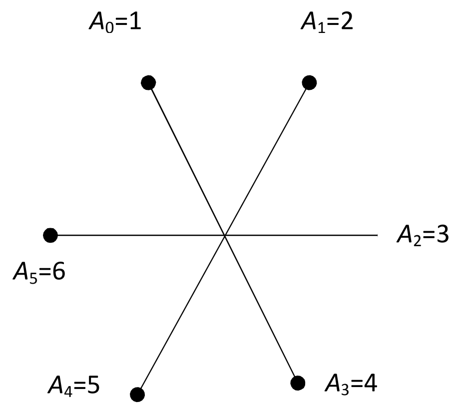

Figure 3.4 *b:* Γ_3 of D_6 on X

Example 3.3

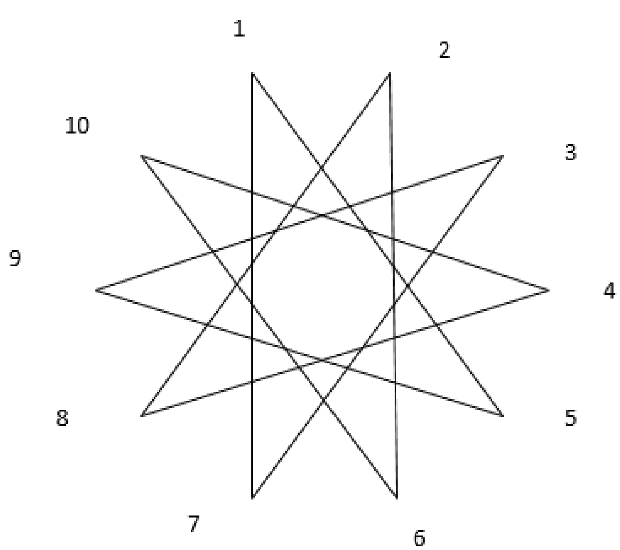

Figure 3.5: The graph Γ_4 of D_{10} on X

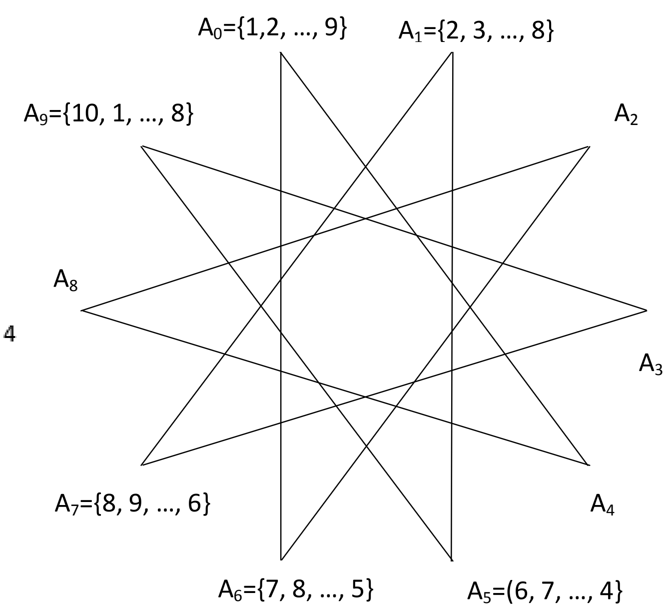

Figure 3.6: The graph Γ_4 of D_{10} on $X^{(9)}$

4. Conclusion

The following theorem suffices as a conclusion of the results.

Theorem 3.16

The action of D_n on X is equivalent to the action of D_n on $X^{(n-1)}$.

Proof:

Using Definition 2.6, let (G_1, X) and $(G_2, X^{(n-1)})$ be the action of D_n on X and the action of D_n on $X^{(n-1)}$, respectively. Define ϕ: $G_1 \rightarrow G_2$ such that $\phi(g)=g$, for all $g \in G_1$. Define the map Θ, through G_x. Now, Θ: $X \rightarrow X^{(n-1)}$ such that $\Theta(x)=X|x$, for all $x \in X$. Now, $\Theta(gx)= X|gx=g(X|x)= \phi(g)\ \Theta(x)$. □

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